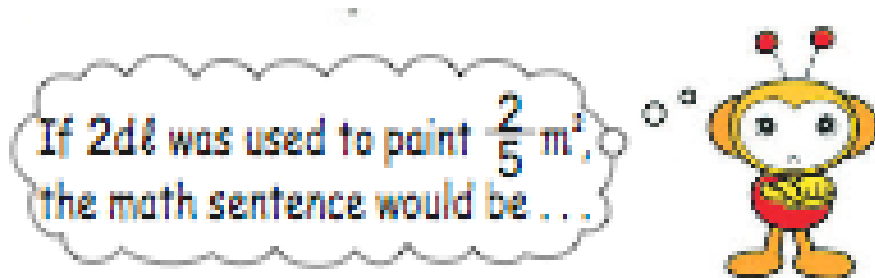


Our IS to reason why, NOT just invert and multiply!

Representing and understanding multiplication and division of fractions



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Keenesaw State University, GA

Goals of this session

- Explore how Japanese textbooks teach multiplication and division of fractions through use of problems and representations.
- Think about how we can use students' prior understanding of multiplication and division to help them understand the multiplication and division of fractions.

NCTM Focal Points

GRADE 6

- **Number and Operations: Developing an understanding of and fluency with multiplication and division of fractions and decimals.** Students use the meanings of fractions, multiplication and division, and the inverse relationship between multiplication and division to make sense of procedures for multiplying and dividing fractions and explain why they work.

Singapore's Mathematics Primary Syllabus (2007)

- Although students should become competent in the various mathematical skills, over-emphasizing procedural skills without understanding the underlying mathematical principles should be avoided.

Japanese Course of Study (1)

- The meanings of (fraction) \times (whole number) and (fraction) \div (whole number) are explained using the same idea used in multiplication and division of whole numbers.

– Elementary School Teaching Guide for the Japanese Course of Study (2006, www.globaledresources.com)

Japanese Course of Study (2)

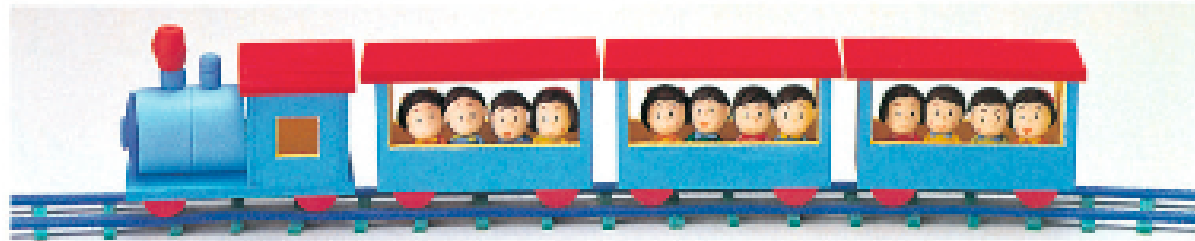
- (fraction) \times (whole number) can be thought of as the calculation of finding how much there is altogether given the size of each unit and the number of units
 - Elementary School Teaching Guide for the Japanese Course of Study (2006, www.globaledresources.com)

Japanese Course of Study (3)

- (fraction) \div (whole number) can be explained by partitive division of whole numbers and by quotative division of whole numbers
 - Elementary School Teaching Guide for the Japanese Course of Study (2006, www.globaledresources.com)

Meaning of Multiplication (1)

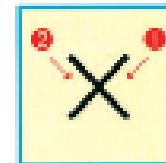
2 How many children are on the train altogether?



There are 4 children in each train car, and there are 3 train cars. So we have 12 children altogether. You can write this using the following math sentence.

$$4 \times 3 = 12$$

“Four multiplied by three equals twelve”

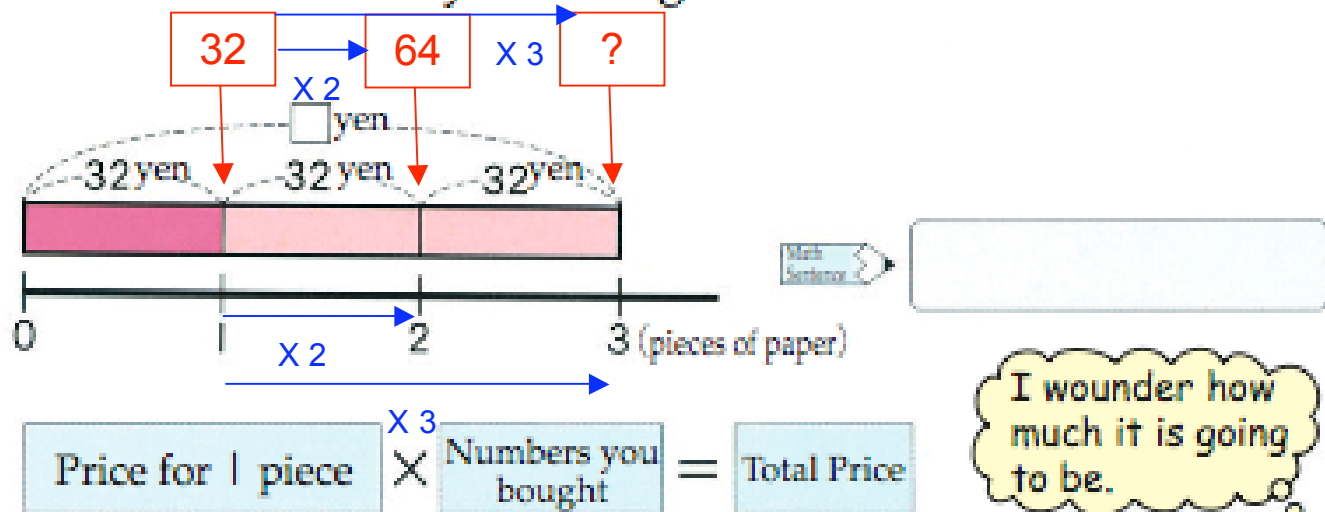


4	×	3	=	12
⋮		⋮		⋮
number of object in each group		number of groups		total number of objects

Representing multiplication with a model

1 Multiplication of 2-digit Number and 1-digit Number

1 We bought 3 pieces of 32-yen construction paper.
How much did they cost altogether?



? Let's think about a way to calculate!





Partitive Meaning of Division

“equal share”
division

1 Calculating How Many for Each Person

There are 12 cookies. If 3 people divide them evenly, how many cookies will one person get?

Quotative (measurement) division

2 Calculating for How Many People

1 ? There are 12 cookies. If one child gets 3 cookies, how many children can get cookies?

1 Let's use counters to find the answer.

for 1 person
Can still pass out more

for 2 people
Can still pass out more

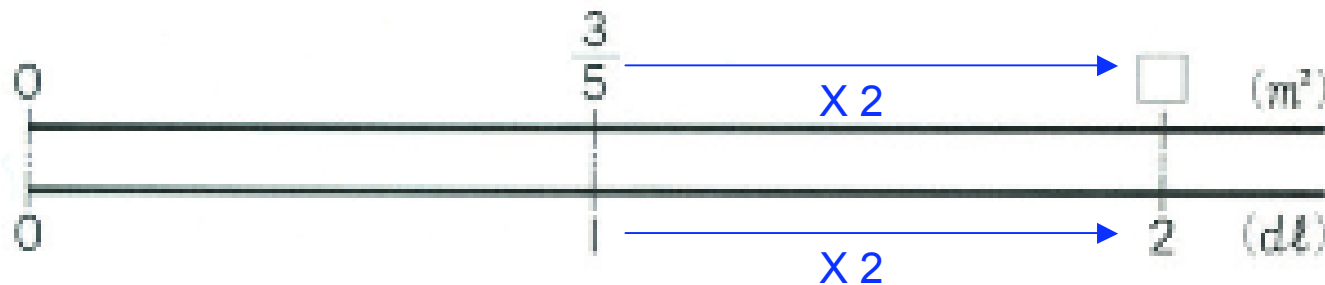
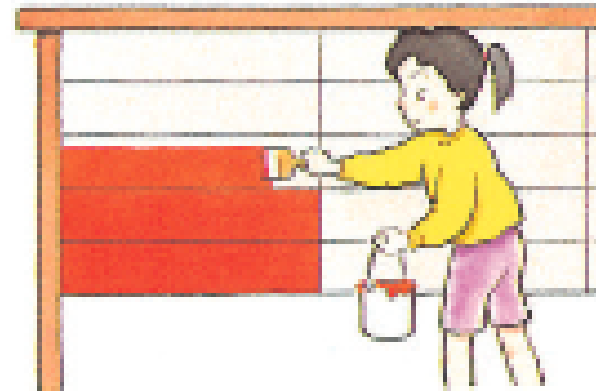
for 3 people
Can still pass out more

for 4 people
All passed out

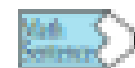
Fraction x Whole Number

1

With 1dl of paint you can paint $\frac{3}{5}\text{m}^2$ of boards.
How many m^2 can you paint with 2dl of paint?

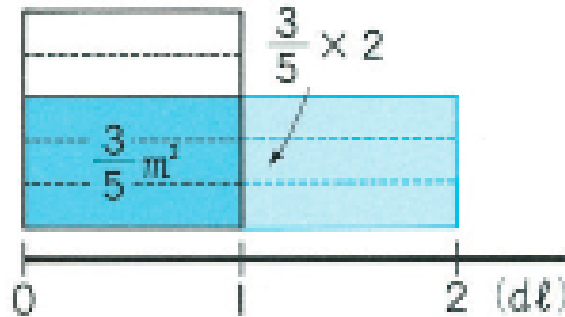


1 What kind of math sentence should you write?



2 Let's think about how to calculate this!

Fraction x Whole Number

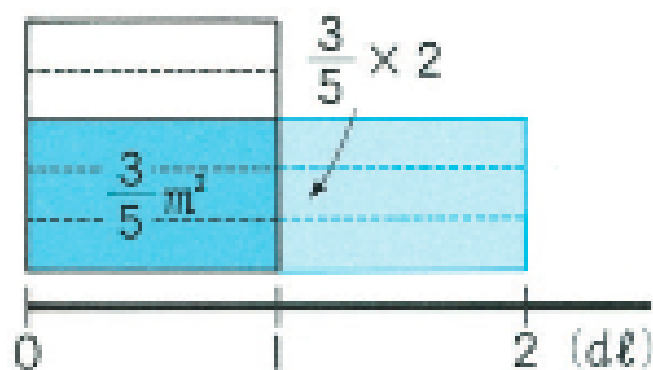


$$\begin{aligned}\frac{3}{5} \times 2 &= \frac{3 \times 2}{5} \\ &= \frac{6}{5} = 1\frac{1}{5} \\ \text{Answer: } &\underline{1\frac{1}{5} \text{ m}^2}\end{aligned}$$

- Related to unitary meaning of fractions
 - The numerator tells the number of unit fractions in the fraction (i.e. $\frac{3}{5}$ means 3 one-fifths)
 - Therefore, you should multiply the numerator by the whole number.
 - It is the same as multiplying whole numbers (3×2)

$$3 \text{ fifths} \times 2 = 6 \text{ fifths}$$

Generalizing the idea of multiplying a fraction by a whole number



$$\begin{aligned}\frac{3}{5} \times 2 &= \frac{3 \times 2}{5} \\ &= \frac{6}{5} = 1\frac{1}{5} \\ \text{Answer: } &1\frac{1}{5} \text{ m}^2\end{aligned}$$



When you multiply a fraction by a whole number, keep the denominator the same and multiply the numerator by the whole number.

$$\frac{b}{a} \times c = \frac{b \times c}{a}$$

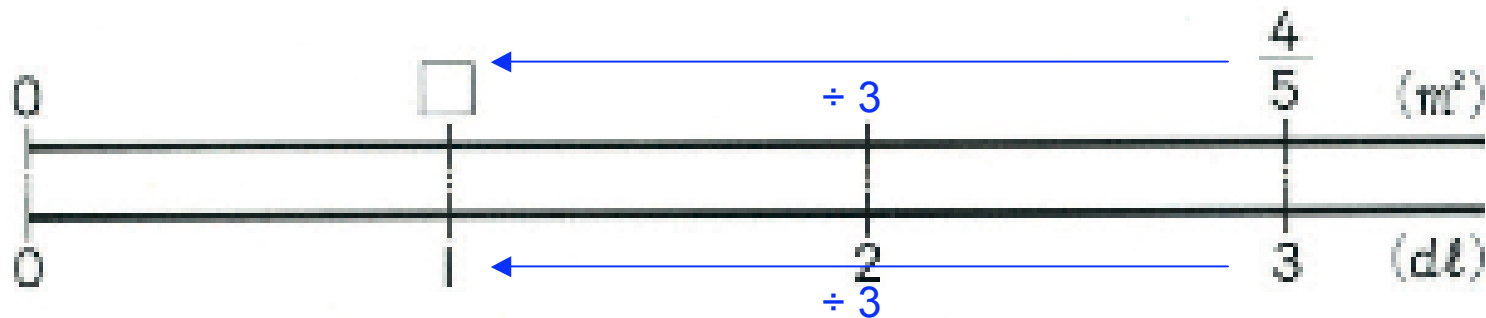
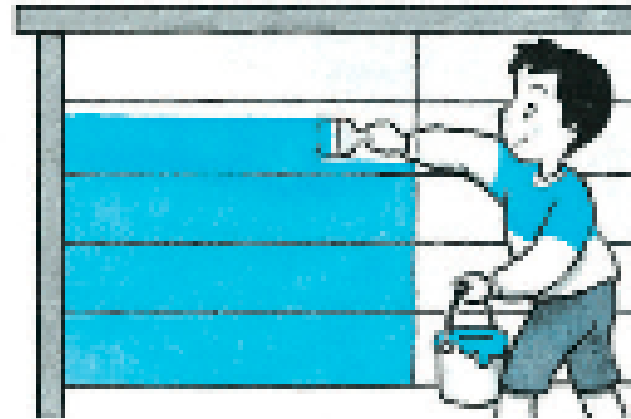
Fraction \div Whole number

2

With 3dℓ of paint you can

paint $\frac{4}{5} \text{ m}^2$ of boards.

How many m^2 can you paint
with 1dℓ of paint?



- 1 What kind of math sentence
should you write?



Partitive division (missing multiplicand)

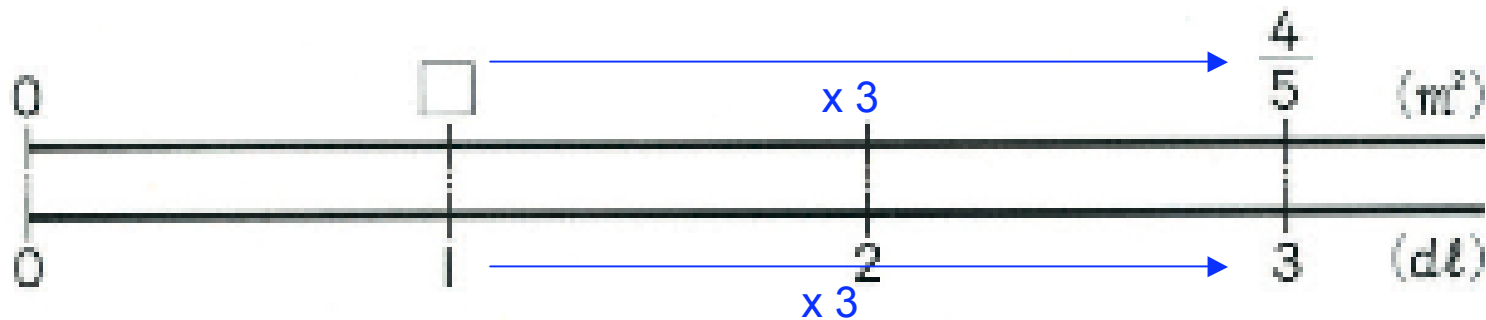
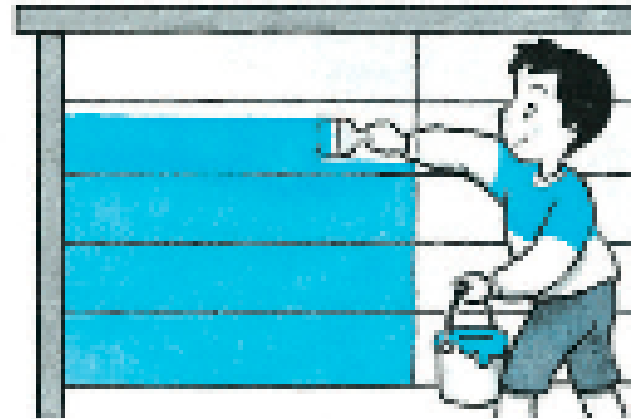
$$\underline{\quad} \times 3 = 4/5$$

2

With 3dℓ of paint you can

paint $\frac{4}{5} \text{ m}^2$ of boards.

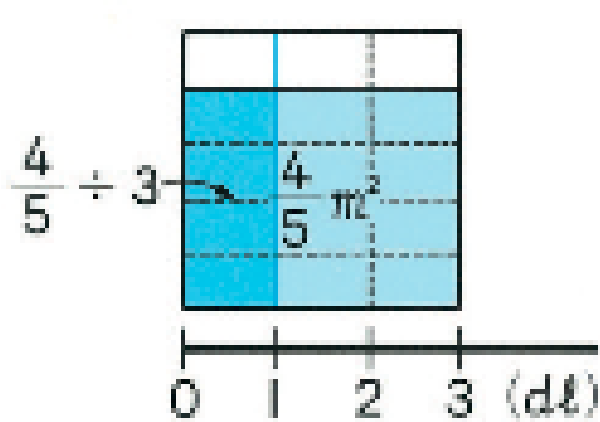
How many m^2 can you paint
with 1dℓ of paint?



- 1 What kind of math sentence
should you write?



Fraction \div Whole number

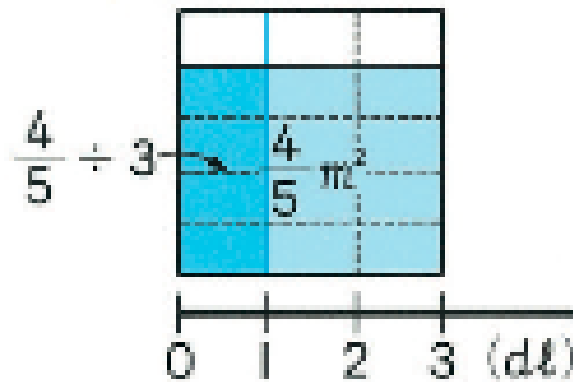


$$\begin{aligned}\frac{4}{5} \div 3 &= \frac{4}{5 \times 3} \\ &= \frac{4}{15}\end{aligned}$$

Answer: $\frac{4}{15} \text{ m}^2$

- Partitive division
 - $\frac{4}{5} \text{ m}^2$ divided into 3 equal parts is easier to think about than the number of 3's in $\frac{4}{5}$.
- Dividing by a whole number affects the denominator
 - The size of the unit fraction (or number of parts in the whole) changed

Generalizing the idea of dividing a fraction by a whole number



$$\begin{aligned}\frac{4}{5} \div 3 &= \frac{4}{5 \times 3} \\ &= \frac{4}{15}\end{aligned}$$

Answer: $\frac{4}{15} m^2$



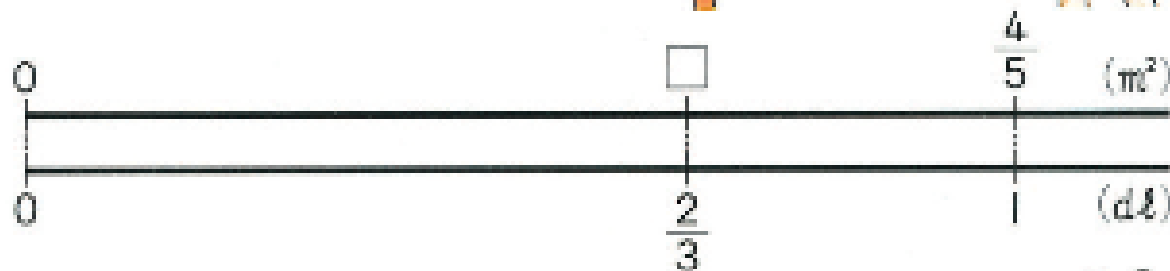
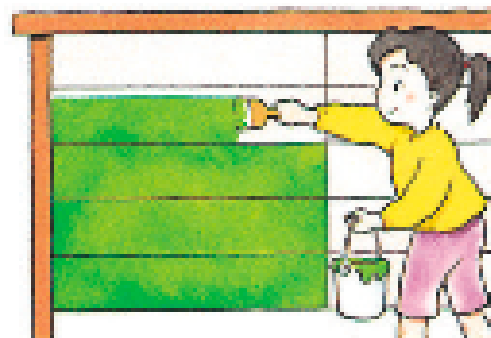
When you divide a fraction by a whole number, keep the numerator the same and multiply the denominator by the whole number.

$$\frac{b}{a} \div c = \frac{b}{a \times c}$$

Fraction x Fraction

1 Multiplying by a Fraction

1 With 1 dl of paint you can paint $\frac{4}{5}\text{ m}^2$ of boards.
How many m^2 can you paint with $\frac{2}{3}\text{ dl}$ of paint?



1 What kind of math sentence should you write?

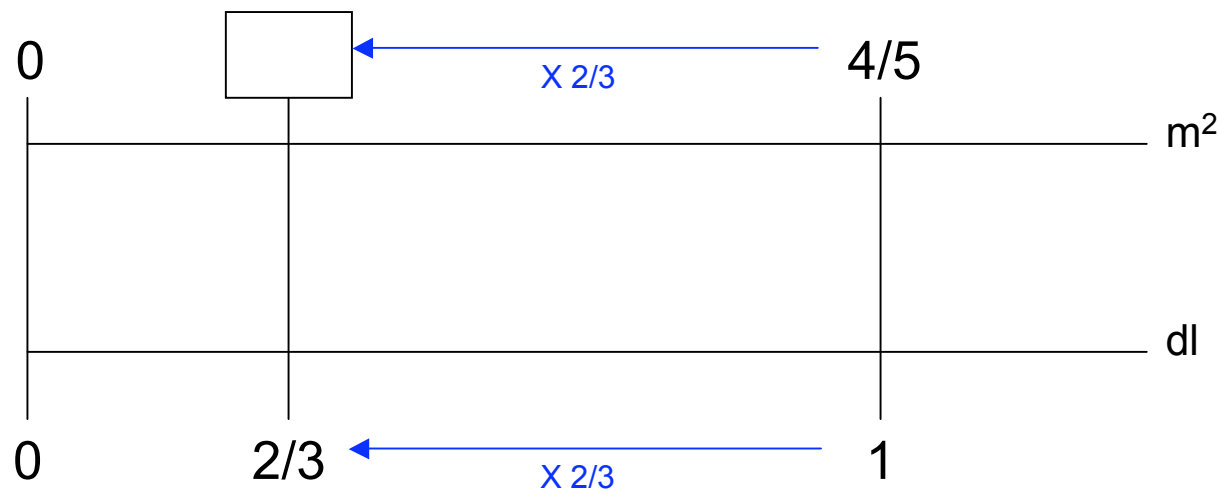
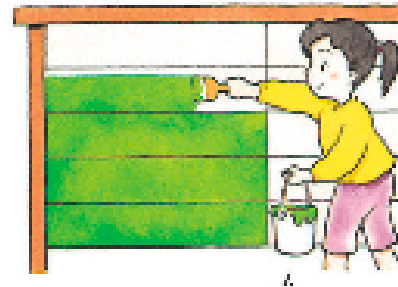
If you use 2 dl of paint, the equation should be . . .



Area that can be painted with 1 dl	\times	Amount you use (in dl)	$=$	Area you can paint
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Creating the double number line model.

With 1 dl of paint you can paint $\frac{4}{5}\text{ m}^2$ of boards.
 How many m^2 can you paint with $\frac{2}{3}\text{ dl}$ of paint?



Area that can be painted with 1 dl	\times	Amount you use (in dl)	$=$	Area you can paint
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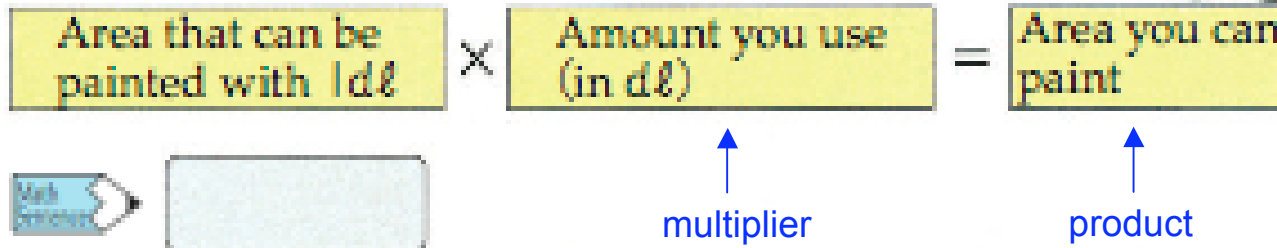
$$\frac{4}{5} \times \frac{2}{3}$$

1 What kind of math sentence should you write?

If you use 2 dl of paint, the equation should be ...

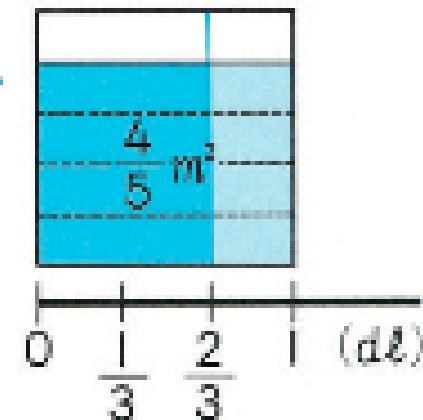


multiplicand



Even though the amount used is expressed as a fraction, to find the area you can paint, multiply in the same way as with whole numbers.

Let's think about how to calculate this!



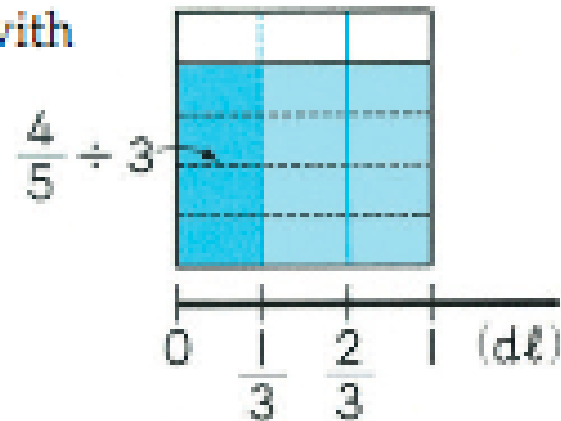
Using an area model with number line to think about the calculation

You can find the area that can be painted with $\frac{2}{3}d\ell$ of paint by first finding the area that can be painted with $\frac{1}{3}d\ell$ and then multiplying it by 2.

2 How many m^2 can you paint with $\frac{1}{3}d\ell$?



Divide $\frac{4}{5}m^2$ into 3 equal parts, and then ...



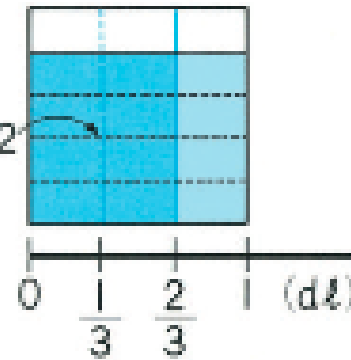
Using an area model with number line to think about the calculation

3 How many m^2 can you paint with $\frac{2}{3} d\ell$?



Multiply the area you can paint with $\frac{1}{3} d\ell$ by 2, and then ...

$$\left(\frac{4}{5} \div 3\right) \times 2$$



$$\frac{4}{5} \times \frac{2}{3} = \left(\frac{4}{5} \div 3\right) \times 2$$

$$= \frac{4}{5 \times 3} \times 2$$

$$= \frac{\boxed{4} \times \boxed{2}}{\boxed{5} \times \boxed{3}} = \boxed{}$$

Answer: $\boxed{} m^2$

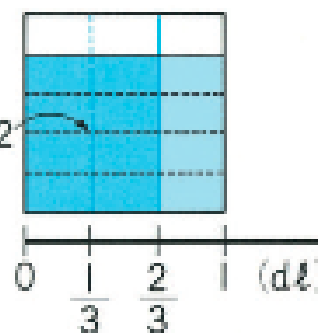
Generalizing the idea of fraction x fraction

3 How many m^2 can you paint with $\frac{2}{3} dl$?



Multiply the area you can paint with $\frac{1}{3} dl$ by 2, and then ...

$$\left(\frac{4}{5} \div 3\right) \times 2$$



$$\frac{4}{5} \times \frac{2}{3} = \left(\frac{4}{5} \div 3\right) \times 2$$

$$= \frac{4}{5 \times 3} \times 2$$

$$= \frac{\square \times \square}{\square \times \square} = \square \quad \text{Answer: } \underline{\square} \text{ m}^2$$

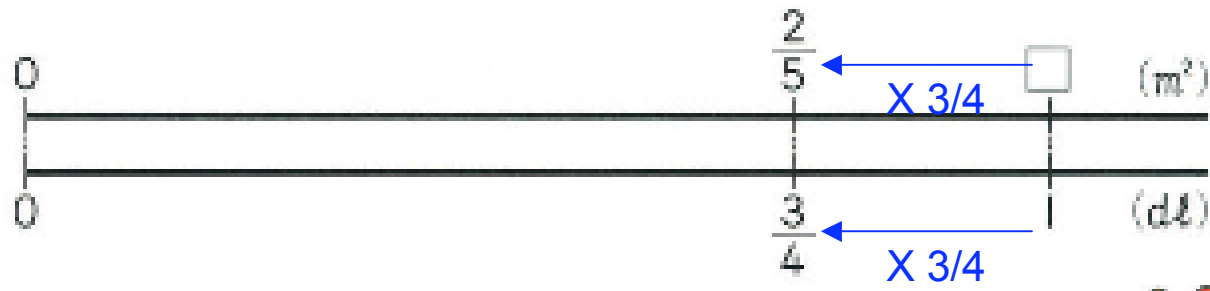
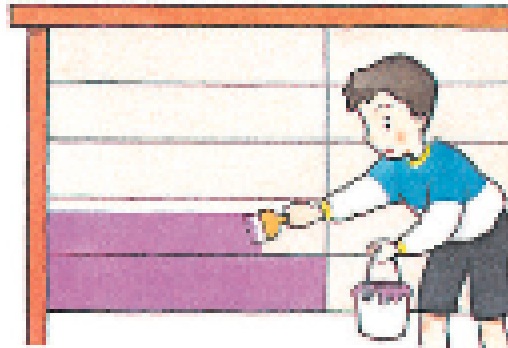
! When you multiply a fraction by another fraction, multiply the two denominators together and the two numerators together.

$$\frac{b}{a} \times \frac{d}{c} = \frac{b \times d}{a \times c}$$

Fraction ÷ Fraction

1 Dividing by a Fraction

1 With $\frac{3}{4}$ dl of paint you can paint $\frac{2}{5}$ m² of board.
How many m² can you paint with 1 dl of paint?



1 What math sentence should we write?

If 2 dl was used to paint $\frac{2}{5}$ m², the math sentence would be ...

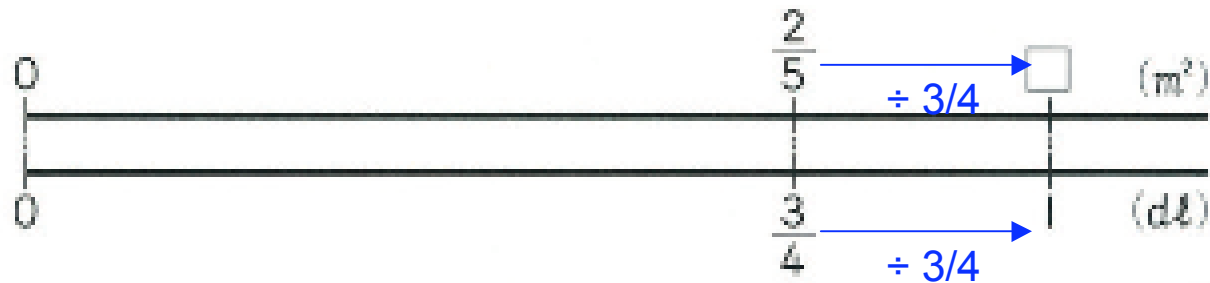
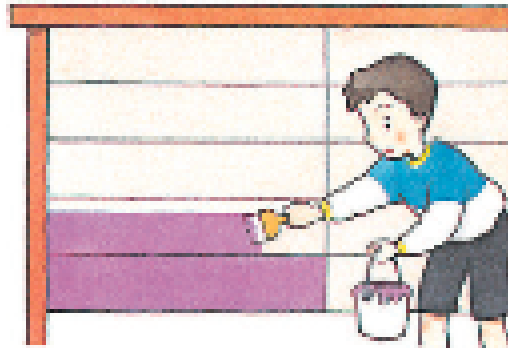


Area painted	÷	Amount used (dl)	=	Area that can be painted with 1 dl
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Fraction ÷ Fraction

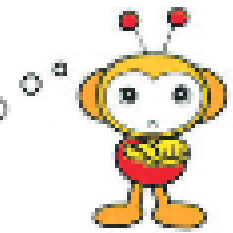
1 Dividing by a Fraction

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How many m² can you paint with 1 dl of paint?



1 What math sentence should we write?

If 2 dl was used to paint $\frac{2}{5}$ m², the math sentence would be ...



Area painted	÷	Amount used (dl)	=	Area that can be painted with 1 dl
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Relating to what was learned before

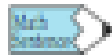
1 What math sentence should we write?

If $2d\ell$ was used to paint $\frac{2}{5} m^2$, the math sentence would be ...



$$\frac{2}{5} \div 2$$

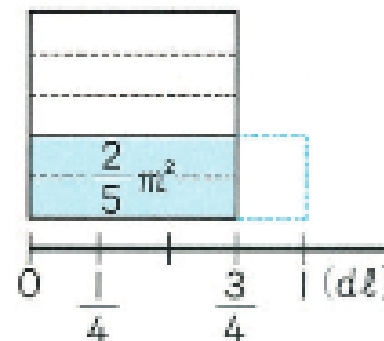
Area painted	\div	Amount used ($d\ell$)	=	Area that can be painted with $1d\ell$
--------------	--------	-------------------------	---	--



$$\frac{2}{5} \div \frac{3}{4}$$

To find the area you can paint with $1d\ell$, you can divide the same way as with whole numbers even though the amount is expressed as fraction.

2 Let's think about how to calculate!



Thinking about the calculation of $\frac{2}{5} \div \frac{3}{4}$

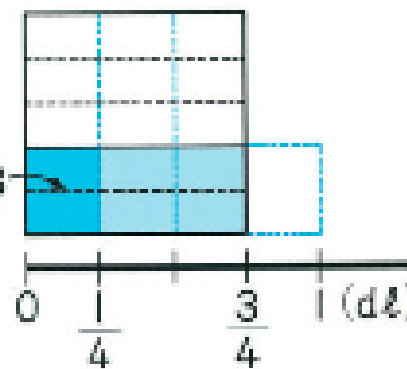
You can calculate the area that can be painted with $1d\ell$ by first finding the area that can be painted with $\frac{1}{4}d\ell$ and then multiplying it by 4.

- 2 How many m^2 can you paint with $\frac{1}{4}d\ell$?



Divide $\frac{2}{5}m^2$ into 3 equal parts and then ...

$$\frac{2}{5} \div 3$$



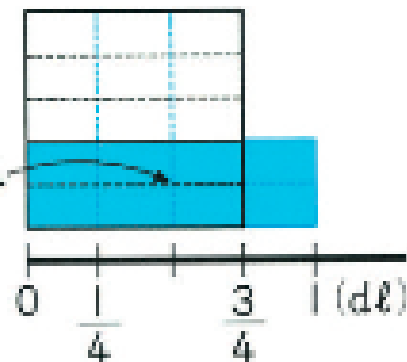
Thinking about the calculation (2)

- 3 How many m^2 can you paint with $1d\ell$?



Multiply the area that you can paint with $\frac{1}{4}d\ell$ of paint by 4, then ...

$$\left(\frac{2}{5} \div 3\right) \times 4$$



$$\frac{2}{5} \div \frac{3}{4} = \left(\frac{2}{5} \div 3\right) \times 4$$

$$= \frac{2}{5 \times 3} \times 4$$

$$= \frac{\boxed{2} \times \boxed{4}}{\boxed{5} \times \boxed{3}} = \boxed{} \quad \text{Answer: } \boxed{} m^2$$


- 4 What fraction can you multiply $\frac{2}{5}$ by to get the same calculation as above?

$\frac{4}{3}$

Generalizing the idea of invert and multiply

$$\begin{aligned}\frac{2}{5} \div \frac{3}{4} &= \left(\frac{2}{5} \div 3 \right) \times 4 \\ &= \frac{2}{5 \times 3} \times 4 \\ &= \frac{\square \times \square}{\square \times \square} = \square \quad \text{Answer: } \underline{\square} \text{ m}^2\end{aligned}$$

- 4 What fraction can you multiply $\frac{2}{5}$ by to get the same calculation as above?

 When you divide by a fraction, invert the denominator and the numerator of the divisor and then multiply the dividend by the resulting new fraction.

$$\begin{aligned}\frac{b}{a} \div \frac{d}{c} &= \frac{b}{a} \times \frac{c}{d} \\ &= \frac{b \times c}{a \times d}\end{aligned}$$

Things to think about

- How can we help students use their prior knowledge of multiplication and division to understand these operations with fractions?
 - It is helpful if the meanings of the factors in the multiplication sentence are clear.
- Context and representations
 - Same context is used in
 - Fraction \times Whole Number
 - Fraction \div Whole Number
 - Fraction \times Fraction
 - Fraction \div Fraction
 - Use of continuous quantities
 - Makes more sense in a real world context
 - Use of models
 - Double number line allows you to use the same model for multiplication and division

Thank you!

- Bill Jackson
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- Tad Watanabe
 - Tad@globaledresources.com
- Helpful web sites
 - www.globaledresources.com
 - www.singaporemath.com